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On the Quantum Mechanical Treatment of Decaying Non-Relativistic Systems

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Abstract

The usual treatment of decaying non-relativistic particles by means of a non-unitary irreducible representation of the Galilei group is deduced from a suitable formulation of symmetry principles. In such a formulation time translation is distinguished from time evolution; this point is crucial to obtain the irreversible behaviour of unstable particles.

1. *Introduction*

In this paper we begin a systematic attempt to put the treatment of irreversible processes in quantum mechanics on a sound conceptual basis. Since the basic description of a physical system is universally accepted to be a reversible one—e.g. a N-body dynamics for a macroscopic system, a field theory for an unstable particle—the problem exists of extracting the description of irreversible processes from this basic reversible theory. In the usual approach to such a problem one introduces more or less justified approximations—truncation of hierarchies, neglect of non-markoffian terms etc.—by which one gets the expected result: e.g. the Boltzmann equation for a dilute gas, the exponential decay law for an unstable particle. However, an unambiguous and fully motivated prescription to obtain irreversible descriptions is so far lacking. Recent developments in the theory of macroscopic systems indicate a way to overcome, in principle, the forementioned difficulties. Prigogine and co-workers (1969) and Balescu & Wallemborn (1971) have proposed and developed a formalism by which independent subdynamics can be extracted from a N-body theory. By a similar formalism, rigorous embedding of an irreversible time evolution into a reversible one has been obtained by the present authors (Lanz *et al.*, 1971); in such a way a concrete formalisation has been

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given of the general concept of embedding an irreversible theory into a reversible one, recently proposed by Ludwig (1970, 1972). Therefore by these results one can hope that the theory of an unstable particle can be linked in a clearcut way to a reversible field theory (Lanz *et al.*, 1973) and, likewise, the theory of a macroscopic system can be rigorously linked to the underlying N-body structure.

In the present paper we are not yet concerned with the embedding problem, but treat the preliminary problem to formulate an irreversible quantum mechanical description for an unstable micro-object. Such a point is not at all trivial from the conceptual point of view, since the usual axiomatics of quantum systems leads to identify the time evolution operator with the unitary time translation operator; in such a way one rules out the possibility of describing a decaying object. In the non-relativistic case one can very easily guess how to introduce a lifetime into the time evolution, i.e. by simply adding a negative imaginary constant to the eigenvalues of the energy: in the relativistic case, however, this problem appears harder (Schulman, 1970) and somewhat paradoxical aspects of the known tentative solutions make a cleaner approach desirable.

In this paper we show that non-unitary time evolution can be naturally reconciled with symmetry under the transformations of the Galilei group if the theory is formulated in a cartesian product space $\mathscr{H} \times \mathscr{R}$ such that, loosely speaking, the time specification of the measurement is given by a point of $\mathscr{R}$ and the remaining specifications correspond to a unit ray in the Hilbert space $\mathscr{H}$. This point is discussed in Section 2. By treating the symmetries under Galilean transformations on this basis, we prove in Sections 3 and 4 that a non-relativistic unstable micro-object is associated to a non-unitary projective (i.e. up to a factor) representation of the Galilei semigroup which, in the usual notations (Inonü & Wigner, 1952), is the set of transformations $(b, \mathbf{a}, \mathbf{v}, R)$ with $b \leqslant 0$. Finally in Section 5 we characterise an unstable particle by a non-unitary projective representation of the Galilei semigroup such that the operators representing the subgroup of spatial rotations and translations, together with the operator representing the position, are an irreducible set of operators. It turns out that the time evolution is described by a Schrödinger equation with a lifetime. It appears that the treatment can be extended in an almost straightforward way to the relativistic case, to yield the result that an unstable relativistic micro-object is associated to a non-unitary representation of the Poincaré semigroup (Schulman, 1970). However, the development of this point is left to a future paper.

2. *Quantum Mechanical Description of an Unstable System*

In the usual axiomatics of quantum mechanics (Von Neumann, 1955) one assumes that to a micro-object M a separable Hilbert-space $\mathscr{H}$ corresponds. A property of M is any set of specifications about M referring either to the way in which M is prepared or to the outcome of an observation on M, including the specification of 'where' and 'when' the preparation or observation happens. To a property γ a subspace $\mathscr{S}_\gamma$ of $\mathscr{H}$ corresponds; γ is said to be sharper than γ' if $\mathscr{S}_\gamma \subseteq \mathscr{S}_{\gamma'}$; therefore γ will be called a maximal sharp property if $\mathscr{S}_\gamma$ is a one-dimensional space. All the properties that we shall consider in the following are intended to be maximal sharp, unless otherwise explicitly stated. Let $\hat{f}_\gamma$ be the unit ray corresponding to γ. Considering for simplicity the case of no super-selection rule, one assumes that conversely for any unit ray $\hat{f}$ at least one γ exists for which $\hat{f}=\hat{f}_\gamma$. If M has by preparation the property γ_0, the probability P_{γ_1,γ_0} that a property γ_1 is ascertained is given by

$$P_{\gamma_1,\gamma_0} = [\hat{f}_{\gamma_1} \cdot \hat{f}_{\gamma_0}]^2 \tag{2.1}$$

where

$$\hat{f}\cdot\hat{g} = |(f,g)|, \qquad f \in \hat{f}, \qquad g \in \hat{g}$$

Let us consider a frame transformation $O \xrightarrow{g^{-1}} O'$ with $g \in \mathscr{G}$, $\mathscr{G}$ being the proper orthochronous Galilei group. g induces a ray transformation $\hat{f}_\gamma \to \hat{f}_{g\gamma} = \hat{U}(g)\hat{f}_\gamma$, where $g\gamma$ is the property which appears to O in the same way as γ appears to O'; $\hat{U}(g)$ is a ray representation of $\mathscr{G}$ such that

$$\hat{U}(g)\hat{f}.\,\hat{U}(g)\hat{g} = \hat{f}.\,\hat{g} \tag{2.2}$$

for every $g \in \mathscr{G}$, $\hat{f}$, $\hat{g}$: see e.g. Wightman (1959). By a theorem of Wigner (1931) a ray correspondence $\hat{f} \xrightarrow{g} \hat{U}(g)\hat{f}$ such that (2.2) holds is induced by a one-to-one linear vector correspondence $f \xrightarrow{g} U(g)f$, $U(g)$ being a unitary projective representation of $\mathscr{G}$. To get an insight into the theory one distinguishes the time specification included in γ, writing

$$\gamma \equiv (\tilde{\gamma}, t) \tag{2.3}$$

and makes the obvious assumption that for any $(\tilde{\gamma}, t)$ a $(\tilde{\gamma}', 0)$ exists such that $P_{(\tilde{\gamma},t),(\tilde{\gamma}',0)} \neq 0$. This implies immediately that the vectors $f_{(\tilde{\gamma},0)}$ span $\mathscr{H}$. Let us consider a time translation of τ and the corresponding unitary operator $U((\tau, \mathbf{0}, \mathbf{0}, I)) = U(\tau)$. Taking into account the meaning of the labels in (2.3) one has

$$f_{(\tilde{\gamma},t)} = U(t) f_{(\tilde{\gamma},0)} \tag{2.4}$$

and by (2.1)

$$P_{(\gamma_1 t_1),(\tilde{\gamma}_0 t_0)} = |(f_{(\gamma_1,0)}, U(-(t_1 - t_0)) f_{(\tilde{\gamma}_0,0)})|^2 \tag{2.5}$$

By equation (2.5) the unitary operator

$$V(\tau) = U(-\tau) \tag{2.6}$$

can be interpreted as a time evolution operator. For any set $\{\tilde{\gamma}_i\}_{i=1,2,\ldots}$ such that $\{f_{(\tilde{\gamma}_i,0)}\}_{i=1,2,\ldots}$, is an orthonormal basis in $\mathscr{H}$ one has that

$$P(t) = \sum_{i=1}^{\infty} P_{(\tilde{\gamma}_i,t),(\tilde{\gamma}_0,t_0)} = 1 \tag{2.7}$$

Equation (2.7) indicates that only a stable system can be described within this formulation. Finally, to characterise a one-particle system one can assume with Mackey (1968) that $\mathscr{H}$ is the space of an irreducible representation for the position operator and the subgroup of $\mathscr{G}$ of rotations and translations.

It seems to us that the crucial point which rules out unstable systems is assumption (2.3). Assumption (2.3) is, however, not justified in the usual formulation of quantum mechanics. Ludwig and collaborators have recently derived such a formulation, in the more general case of mixtures instead of pure states, within a realistic theory of measurement starting from reasonable axioms about the macroscopic systems which enter into experiments concerning micro-objects (Ludwig, 1967, 1970). In their theory the preparation of the micro-object by a macroscopic system and the effects produced by the micro-object on a macroscopic system do not contain one sharp time specification. For example, the effects can consist of a set of macroscopic modifications at different time points; moreover two effects at different time points, which exhibit the same statistics for every preparation, correspond to the same operator (Ludwig, 1972).

On the other hand in order to have a clearly interpretable description of a one-particle system it seems to us convenient to use a formulation in which a sharp time specification can be attached to the prepared or observed properties in a consistent way. We do this representing mathematically the preparation and the observation of a micro-object M by a couple $(\hat{f}_\gamma, t)$, where t refers to the time at which the preparation or the observation is performed and γ to the remaining specifications of the properties; $\hat{f}_\gamma$ is a unit ray in a separable Hilbert space, each unit ray of which is in turn associated to at least one property. It is outside the scope of this paper to give a justification of this more schematic description, starting from general axioms about the set of experiments such that a suitable time specification can be attributed to the preparations and to the observations of a micro-object. Such a set of experiments is a restriction of the full set of experiments considered by Ludwig. Our only claim is that an unstable particle can be consistently described within our scheme.

Let us consider a collection of N_{γ_0, t_0} objects identical to M having by preparation at time t_0 the property γ_0. We assume that the number $N_{\gamma_1, t_1; \gamma_0, t_0}$ of objects, for which the property γ_1 is ascertained at a time $t_1 \geqslant t_0$, is given by

$$N_{\gamma_1, t_1; \gamma_0, t_0} = [\hat{f}_{\gamma_1} \cdot \hat{V}(t_1, t_0) \hat{f}_{\gamma_0}]^2 \cdot N_{\gamma_0, t_0} \tag{2.8}$$

where $\hat{V}(t_1, t_0)$ is a contractive transformation of the rays† in $\mathscr{H}$, depending on two parameters t_0, t_1, with $-\infty < t_0 < +\infty$, $t_1 \geqslant t_0$; we assume that the domain of $\hat{V}(t_1, t_0)$ is the set of all rays in $\mathscr{H}$ and that

$$\hat{V}(t_0, t_0) = \hat{I}, \quad -\infty < t_0 < +\infty, \quad \hat{V}(t_1, t_0) \neq \hat{O} \tag{2.9}$$

† A ray $\hat{f}$ in $\mathscr{H}$ is the set of all vectors of the form ωf, where f is a fixed vector of $\mathscr{H}$ and ω any complex number with $|\omega| = 1$. We define $\alpha\hat{f} = \widehat{\alpha f}$ for any complex α. We call (somewhat improperly) norm of $\hat{f}$ the number $\|f\|$ and indicate it by $\|\hat{f}\|$. A ray transformation $\hat{A}$ is called contractive if $\|\hat{A}\hat{f}\| \leqslant \|\hat{f}\|$, norm conserving if $\|\hat{A}\hat{f}\| = \|\hat{f}\|$. The strong continuity of a ray transformation is defined with respect to the distance $d(\hat{f}, \hat{g}) = \underset{f \in \hat{f}, g \in \hat{g}}{\text{Min}} \|f - g\|$.

where $\hat{I}$ is the identity operator of the rays in $\mathscr{H}$, and $\hat{O}$ the null operator. If $\hat{V}(t_1, t_0)$ is norm conserving, M is stable; if $\hat{V}(t_1, t_0)$ is strictly contractive, M is unstable. We stress that $\hat{V}(t_1, t_0)$ has no connection with time translations. $N_{\gamma_1, t_1; \gamma_0, t_0}$ is intended to be a theoretical prevision, which is expected to fit the experimental values $N^{\exp}_{\gamma_1, t_1; \gamma_0, t_0}$ when N_{γ_0, t_0} is large enough.

Equation (2.8) is the basic axiom in our formulation, as well as (2.1) was in the previous one. Contrary to (2.1), it is not an axiom on probabilities, since using the sharp time specification it is natural to refer probabilities to the collection of systems present at time t. From (2.8) the number of systems of the collection at time t_1 is given by

$$\begin{aligned} N_{\gamma_0, t_0}(t_1) &= \sum_{i=1}^{\infty} [\hat{f}_{\gamma_i} \cdot \hat{V}(t_1, t_0) \hat{f}_{\gamma_0}]^2 N_{\gamma_0, t_0} \\ &= \| \hat{V}(t_1, t_0) \hat{f}_{\gamma_0} \|^2 \cdot N_{\gamma_0, t_0} \end{aligned} \tag{2.10}$$

where $\{\gamma_i\}_{i=1,2,\ldots}$ is any set of properties such that, choosing arbitrarily $f_{\gamma_i} \in \hat{f}_{\gamma_i}$, the set of elements $\{f_{\gamma_i}\}_{i=1,2,\ldots}$ is an orthonormal basis in $\mathscr{H}$. When M is stable, $N_{\gamma_0, t_0}(t_1)$ is independent of t_1 so that (2.8) becomes an axiom about the probabilities $N_{\gamma_1, t_1; \gamma_0, t_0}/N_{\gamma_0 t, 0}$.

In the general case one can define a probability

$$P_{\gamma_0, t_0}(\gamma_1, t_1) = \frac{N_{\gamma_1, t_1; \gamma_0, t_0}}{N_{\gamma_0, t_0}(t_1)} = [\hat{f}_{\gamma_1} \cdot \hat{f}_{\gamma_0, t_0}(t_1)]^2 \tag{2.11}$$

with

$$\hat{f}_{\gamma_0, t_0}(t_1) = \frac{\hat{V}(t_1, t_0) \hat{f}_{\gamma_0}}{\| \hat{V}(t_1, t_0) \hat{f}_{\gamma_0} \|} \tag{2.12}$$

where $P_{\gamma_0, t_0}(\gamma_1, t_1)$ gives the probability of finding the property γ_1 at time t_1 in the collection of $N_{\gamma_0, t_0}(t_1)$ systems. The unit ray $\hat{f}_{\gamma_0, t_0}(t_1)$ is a generalisation of the usual 'state vector', to which (2.12) reduces when M is stable.

In conclusion, our formulation (2.8) is certainly more schematic than (2.1), but allows a clean physical interpretation and an easy introduction of definite physical assumptions as the existence of a position operator for a Galilean particle, etc. In Lanz *et al.* (1973) we shall discuss the embedding of a micro-object, described in the present paper, into a field theoretical model. In that context formulation (2.1) provides itself naturally for the model, since, the properties of the model do not need a concrete physical interpretation.

3. *Galilean Invariance*

Due to the symmetry of the theory under the transformations $g \in \mathscr{G}$, by an obvious generalisation of the usual procedure (Wightman, 1959) one has a representation $\mathscr{U}(g)$ of $\mathscr{G}$ on the space of the couples $(\hat{f}, t)$, $\hat{f}$ unit ray of $\mathscr{H}$, $t \in R$. Taking into account the meaning of the index t in $(\hat{f}, t)$ one has immediately for every $g = (b, \mathbf{a}, \mathbf{v}, R)$

$$\mathscr{U}((b, \mathbf{a}, \mathbf{v}, R))(\hat{f}, t) = (\hat{U}(\mathbf{a}, \mathbf{v}, R; t)\hat{f}, t + b) \tag{3.1}$$

where $\hat{U}(\mathbf{a}, \mathbf{v}, R; t)$ is an operator on the unitary rays; the standard notation for g is used: b time translation, $\mathbf{a}$ space translation, $\mathbf{v}$ acceleration, R rotation. We assume that $\hat{U}(\mathbf{a}, \mathbf{v}, R; t)$ depends continuously on $\mathbf{a}$, $\mathbf{v}$, R. From

$$\mathscr{U}(g_2) . \mathscr{U}(g_1) = \mathscr{U}(g_2 . g_1)$$

taking into account that

$$(b_2, \mathbf{a}_2, \mathbf{v}_2, R_2)(b_1, \mathbf{a}_1, \mathbf{v}_1, R_1) \\ = (b_2 + b_1, R_2\mathbf{a}_1 + \mathbf{v}_2 b_1 + \mathbf{a}_2, R_2\mathbf{v}_1 + \mathbf{v}_2, R_2 R_1)$$

one has by (3.1)

$$\hat{U}(\mathbf{a}_2, \mathbf{v}_2, R_2; t + b_1)\hat{U}(\mathbf{a}_1 \mathbf{v}_1, R_1; t) \\ = \hat{U}(R_2\mathbf{a}_1 + \mathbf{v}_2 b_1 + \mathbf{a}_2, R_2\mathbf{v}_1 + \mathbf{v}_2, R_2 R_1; t) \tag{3.2}$$

Such a relation leads in particular to

$$\hat{U}(\mathbf{a}, \mathbf{v}, R; t) = \hat{U}(\mathbf{a} + \mathbf{v}t, \mathbf{v}, R; 0) \tag{3.3}$$

and to

$$\hat{U}(\mathbf{a}_2, \mathbf{v}_2; R_2; t)\hat{U}(\mathbf{a}_1, \mathbf{v}_1, R_1; t) \\ = \hat{U}(R_2\mathbf{a}_1 + \mathbf{a}_2, R_2\mathbf{v}_1 + \mathbf{v}_2, R_2 R_1; t) \tag{3.4}$$

Equation (3.4) means that $\hat{U}(\mathbf{a}, \mathbf{v}, R; t)$ is a ray representation of the subgroup $\mathscr{G}_0$ corresponding to $b = 0$, for any fixed t; equation (3.3) links representations with different t. The condition of Galilean invariance of the description given by the numbers $N_{\gamma_1, t_1; \gamma_0, t_0}$ is finally expressed by the relation

$$\hat{f} . \hat{V}(t_1, t_0)\hat{g} = \hat{U}(\mathbf{a}, \mathbf{v}, R; t_1)\hat{f} . \hat{V}(t_1 + b, t_0 + b)\hat{U}(\mathbf{a}, \mathbf{v}, R; t_0)\hat{g} \tag{3.5}$$

Taking $t_1 = t_0 = t$, by (2.9) equation (3.5) becomes

$$\hat{f} . \hat{g} = \hat{U}(\mathbf{a}, \mathbf{v}, R; t)\hat{f} . \hat{U}(\mathbf{a}, \mathbf{v}, R; t)\hat{g} \tag{3.6}$$

Then by the theorem of Wigner (1931) the ray correspondence $\hat{U}(\mathbf{a}, \mathbf{v}, R; t)$ is induced by a unitary projective representation $U(\mathbf{a}, \mathbf{v}, R; t)$ of $\mathscr{G}_0$:

$$U(\mathbf{a}_2, \mathbf{v}_2, R_2; t)U(\mathbf{a}_1, \mathbf{v}_1, R_1; t) \\ = \omega(\mathbf{a}_2, \mathbf{v}_2, R_2, \mathbf{a}_1, \mathbf{v}_1, R_1; t)U(R_2\mathbf{a}_1 + \mathbf{a}_2, R_2\mathbf{v}_1 + \mathbf{v}_2, R_2 R_1; t) \tag{3.7}$$

where $\omega(\mathbf{a}_2, \mathbf{v}_2, R_2, \mathbf{a}_1, \mathbf{v}_1, R_1; t)$ is a suitable phase factor. By equation (3.3) one can take

$$U(\mathbf{a}, \mathbf{v}, R; t) = U(\mathbf{a} + \mathbf{v}t, \mathbf{v}, R; 0) \tag{3.8}$$

By equation (3.5), in the case $\mathbf{a} = \mathbf{v} = 0$, $R = I$, one has

$$\hat{V}(t_2, t_1) = \hat{V}(t_2 + b, t_1 + b) = \hat{V}(t_2 - t_1) \tag{3.9}$$

These are the consequences of symmetry considerations.

4. *The Objectivity Requirement*

We assume that the contractive ray transformation $\hat{V}(t)$, $t \geqslant 0$, is induced by a linear operator $V(t)$ on $\mathscr{H}$, depending on t in a strongly continuous way for $t < \varepsilon$, ε arbitrarily small.† $V(t)$ is then a linear contractive operator on $\mathscr{H}$ for any $t \geqslant 0$. Without any loss of generality we can take, by (2.9), $V(0) = I$.

Equation (3.5), by (3.8) and (3.9), can be rewritten in the following way:

$$V(t) = \omega(\mathbf{a}, \mathbf{v}, R; t)\, U^{-1}(\mathbf{a} + \mathbf{v}t, \mathbf{v}, R; 0) \\ .\, V(t)\, U(\mathbf{a}, \mathbf{v}, R; 0), \qquad t \geqslant 0 \tag{4.1}$$

where $\omega(\mathbf{a}, \mathbf{v}, R; t)$ is a suitable phase factor.

Now, since the system M is described by means of $\hat{f}_{\gamma_0, t_0}(t)$ and $N_{\gamma_0, t_0}(t)$, we require that an 'objective' character be given to such a description. More specifically, we make the following fundamental *objectivity requirement*: at any time t_1 the observer must be able to ascertain at least one property on a collection of objects M (prepared at an earlier time t_0) without perturbing the time evolution of the collection for $t > t_1$. In fact, under this condition we can say that M 'has' such a property at time t_1. It is immediately seen that this requirement is satisfied if $V(t)$ is an up-to-a-factor one-parameter semigroup:

$$V(t_1)\, V(t_2) = \omega(t_1, t_2)\, V(t_1 + t_2), \qquad t_1, t_2 \geqslant 0 \tag{4.2}$$

where $\omega(t_1, t_2)$ is a suitable phase factor. In fact, one has in such a case from (2.8) and (2.12) that

$$|(f_{\gamma_2}, V(t - t_0) f_{\gamma_0})|^2 N_{\gamma_0, t_0}(t_0) = |(f_{\gamma_2}, V(t - t_1) f_{\gamma_1})|^2 N_{\gamma_1, t_1}(t_1) \tag{4.3}$$

with

$$f_{\gamma_1} = f_{\gamma_0, t_0}(t_1), \qquad N_{\gamma_1, t_1}(t_1) = N_{\gamma_0, t_0}(t_1), \qquad t \geqslant t_1 \geqslant t_0$$

† One would prefer to assume strong continuity of $\hat{V}(t)$ with respect to the distance $d(\hat{f}, \hat{g}) = \min_{f \in \hat{f}, g \in \hat{g}} \|f - g\|$ for $t < \varepsilon$, ε arbitrarily small. Then assuming that $\hat{V}(t)$ is induced by a linear operator $V'(t)$, one should look for a modulus one function $\omega(t)$ such that $V(t) = \omega(t)\, V'(t)$ is strongly continuous for $t < \varepsilon'$. In the case of a norm conserving transformation $V(t)$, by a theorem of Bargmann (1954) such $\omega(t)$ always exists; in the case of a strictly contractive $\hat{V}(t)$, the same result can be obtained with the further assumption that

$$\arg\{(V'(t) f, V'(t) g)\}$$

is a continuous functional of t for $t < \varepsilon$, $\forall\, f, g \in \mathscr{H}$, and that the continuity in t of

$$|(V'(t) f, V'(t) g)|$$

is uniform with respect to f and g, with $\|f\| = \|g\| = 1$.

In words, if γ_1 is a property such that $f_{\gamma_0, t_0}(t_1) = f_{\gamma_1}$, the ascertainment of γ_1 at time t_1 does not perturbate the collection. Conversely, from the condition that for any couple of unit vectors f and g and any $t \geqslant t_1 \geqslant t_0$ a unit vector g' and a number N' exist, such that

$$|(f, V(t-t_0)g)|^2 N = |(f, V(t-t_1)g')|^2 N' \tag{4.4}$$

assuming that g' depends linearly on g and is independent of t, one derives that $V(t)$ is an up-to-a-factor one-parameter semigroup and that

$$\frac{N}{N'} = \|V(t_1 - t_0)g\|^2 \tag{4.5}$$

In fact (4.4) implies that

$$\sqrt{(N)}\, V(t-t_0)g = \omega(t-t_0, t-t_1)\sqrt{(N')}\, V(t-t_1)g' \tag{4.6}$$

$\omega(t-t_0, t-t_1)$ being a phase factor. Putting $t = t_1$ in (4.6) and taking into account that $V(0) = I$, one has

$$\sqrt{(N)}\, V(t_1 - t_0)g = \omega(t_1 - t_0, 0)\sqrt{(N')}\, g' \tag{4.7}$$

Equation (4.5) follows immediately from (4.7). Substituting (4.7) into (4.6) one has

$$V(t-t_0) = \omega'(t-t_1, t_1-t_0)\, V(t-t_1)\, V(t_1-t_0) \tag{4.8}$$

Summarising the results so far, we have from (3.7), (4.1) and (4.2) that the set of operators

$$\tilde{U}(b, \mathbf{a}, \mathbf{v}, R) = V(-b)\, U(\mathbf{a}, \mathbf{v}, R; 0) \tag{4.9}$$

where $b \leqslant 0$ and $\mathbf{a}$, $\mathbf{v}$, R vary in the usual range, give a (in general non-unitary) projective representation of the Galilei semigroup. In fact:

$$\begin{aligned} &\tilde{U}(b_2, \mathbf{a}_2, \mathbf{v}_2, R_2)\, \tilde{U}(b_1, \mathbf{a}_1, \mathbf{v}_1, R_1) \\ &\quad = \tilde{\omega}(b_1, \mathbf{a}_1, \mathbf{v}_1, R_1, b_2, \mathbf{a}_2, \mathbf{v}_2, R_2) \\ &\quad \times \tilde{U}(b_2 + b_1, R_2\mathbf{a}_1 + \mathbf{v}_2 b_1 + \mathbf{a}_2, R_2\mathbf{v}_1 + \mathbf{v}_2, R_2 R_1) \end{aligned} \tag{4.10}$$

where $\tilde{\omega}(b_1, \mathbf{a}_1, \mathbf{v}_1, R_1, b_2, \mathbf{a}_2, \mathbf{v}_2, R_2)$ is a suitable modulus one factor. In the following we shall indicate, as is usual,

$$\begin{aligned} \tilde{U}(0, \mathbf{a}, \mathbf{0}, I) &= T(\mathbf{a}), & \mathbf{a} &\in R_3 \\ \tilde{U}(0, \mathbf{0}, \mathbf{v}, I) &= G(\mathbf{v}), & \mathbf{v} &\in R_3 \\ \tilde{U}(0, \mathbf{0}, \mathbf{0}, R) &= O(R), & R &\in SO(3) \end{aligned} \tag{4.11}$$

5. *Characterisation of a Particle*

An obvious unsharp property for a micro-object is its presence in a certain region E of space. Such a fact can be formalised, following Mackey

(1968), introducing a projection-valued measure $P(E)$, defined on the Borel σ-algebra of R_3, satisfying the following conditions:

$$\begin{aligned} T(\mathbf{a})P(E)T(\mathbf{a})^{-1} &= P(t(a)E) \\ O(R)P(E)O(R)^{-1} &= P(o(R)E) \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} t(\mathbf{a})E &\equiv \{\mathbf{x}: \mathbf{x}-\mathbf{a}\in E\} \\ o(R)E &\equiv \{\mathbf{x}: R^{-1}\mathbf{x}\in E\} \end{aligned}$$

To characterise a particle we assume that the set of operators $T(\mathbf{a})$, $O(R)$, $P(E)$ for all $\mathbf{a}$, R, E is irreducible on $\mathscr{H}$. Therefore one has to analyse the problem to find a projective representation of the Galilei semigroup such that $P(E)$ can be defined according to (5.1) and $T(\mathbf{a})$, $O(R)$, $P(E)$ are an irreducible set of operators. As is well known such a problem has been solved in the case of a stable particle, in which one looks for a unitary projective representation of $\mathscr{G}$, see Mackey (1968). We shall give in this section, in a sketchy way, the generalisation of this result to the case of an unstable particle.

As for the multiplication rules of the unitary operators representing the elements of $\mathscr{G}_0$, we have

$$T(\mathbf{a}_2)T(\mathbf{a}_1) = T(\mathbf{a}_2+\mathbf{a}_1) \tag{5.2a}$$

$$O(R_2)O(R_1) = O(R_2R_1) \tag{5.2b}$$

$$O(R)T(\mathbf{a}) = T(R\mathbf{a})O(R) \tag{5.2c}$$

$$G(\mathbf{v}_2)G(\mathbf{v}_1) = G(\mathbf{v}_2+\mathbf{v}_1) \tag{5.2d}$$

$$O(R)G(\mathbf{v}) = G(R\mathbf{v})O(R) \tag{5.2e}$$

$$T(\mathbf{a})G(\mathbf{v}) = \mathrm{e}^{-iM\mathbf{v}.\mathbf{a}}G(\mathbf{v})T(\mathbf{a}) \tag{5.2f}$$

where the factors are chosen according to Bargmann (1954) and the covering group $SU(2)$ of $SO(3)$ is considered; M is a real number. It is convenient to introduce the unitary group.

$$Z(\mathbf{d}) = \int_{R_3} \mathrm{e}^{i\mathbf{x}.\mathbf{d}}\,dP, \qquad \mathbf{d}\in R_3 \tag{5.3}$$

where $P(E)$ is the projection-valued measure introduced in (5.1). One can call 'position operator' the generator of $Z(\mathbf{d})$ times $(-i)$.

We have immediately, from (5.1), that

$$T(\mathbf{a})Z(\mathbf{d}) = \mathrm{e}^{-i\mathbf{a}.\mathbf{d}}Z(\mathbf{d})T(a) \tag{5.4a}$$

$$O(R)Z(\mathbf{d}) = Z(R\mathbf{d})O(R) \tag{5.4b}$$

The problem of finding the unitary irreducible representations of the multiplication rules (5.2a, b, c) and (5.4a, b) has been solved by Mackey (1968). Any such representation is in correspondence with an irreducible unitary representation of $SU(2)$ and therefore is labelled by an integer or half-integer index j. The Hilbert space $\mathscr{H}$ is the direct integral of Hilbert spaces $\mathfrak{H}(\mathbf{p})$, $\mathbf{p} \in R_3$ with the Lebesgue measure, $\mathfrak{H}(\mathbf{p})$ being the space $\mathfrak{H}^j$ of the irreducible representation $\mathscr{D}^j$ of $SU(2)$. Furthermore one has for every $\{f(\mathbf{p})\} \in \mathscr{H}$

$$Z(\mathbf{d})\{f(\mathbf{p})\} = \{f(\mathbf{p}-\mathbf{d})\} \tag{5.5a}$$

$$O(R)\{f(\mathbf{p})\} = \{D^j(R)\, f(R^{-1}\mathbf{p})\} \tag{5.5b}$$

$$T(\mathbf{a})\{f(\mathbf{p})\} = \{\mathrm{e}^{-i\mathbf{a}\mathbf{p}} f(\mathbf{p})\} \tag{5.5c}$$

where $D^j(R)$ is the operator representing R in $\mathscr{D}^j$. Since the multiplication rules between $T(\mathbf{a})$, $O(R)$, $G(\mathbf{v})$ are identical to those between $T(\mathbf{a})$, $O(R)$, $Z(M\mathbf{v})$ one has by the irreducibility of the representation of such multiplication rules that a unitary operator S on $\mathscr{H}$ exists such that

$$T(\mathbf{a}) = ST(\mathbf{a})\, S^{-1} \tag{5.6a}$$

$$O(R) = SO(R)\, S^{-1} \tag{5.6b}$$

$$G(\mathbf{v}) = SZ(M\mathbf{v})\, S^{-1} \tag{5.6c}$$

By (5.5c) and (5.6a) we have

$$S\{f(\mathbf{p})\} = \{\sigma(\mathbf{p})\, f(\mathbf{p})\} \tag{5.7}$$

where $\sigma(\mathbf{p})$ is a unitary linear operator on $\mathfrak{H}^j$. By the irreducibility of the set (05.5) and by equation (4.1) one easily sees that the kernel of $V(t)$ contains only the zero vector of $\mathscr{H}$ and that the range of $V(t)$ is $\mathscr{H}$, so that $V^{-1}(t)$ is bounded. Then, defining for $t \leqslant 0$ $V(t) = V^{-1}(-t)$ one obtains by (4.9) a projective representation of the Galilei group which in general is not unitary.

The factor in equation (4.2) can be eliminated; this follows by a trivial generalisation of a theorem of Bargmann about projective representations of one-parameter groups (Bargmann, 1954). We have therefore

$$V(t_1)\, V(t_2) = V(t_1 + t_2), \qquad t_1, t_2 \in R; \qquad V(0) = I \tag{5.8}$$

As for the factors in the multiplication rules between $V(t)$ and the operators representing $\mathscr{G}_0$ one has easily

$$T(\mathbf{a})\, V(t) = V(t)\, T(\mathbf{a}) \tag{5.9a}$$

$$O(R)\, V(t) = V(t)\, O(R) \tag{5.9b}$$

$$T(\mathbf{v}t)\, G(\mathbf{v})\, V(t) = \mathrm{e}^{-i1/2M\mathbf{v}^2 t}\, V(t)\, G(\mathbf{v}) \tag{5.9c}$$

Introducing the one-parameter group

$$V'(t) = S^{-1}\, V(t)\, S, \qquad t \in R \tag{5.10}$$

one has by (5.6) and (5.9) that

$$T(\mathbf{a})\, V'(t) = V'(t)\, T(\mathbf{a}) \tag{5.11a}$$

$$O(R)\, V'(t) = V'(t)\, O(R) \tag{5.11b}$$

$$T(\mathbf{v}t)\, Z(M\mathbf{v})\, V'(t) = \mathrm{e}^{-i1/2Mv^2 t}\, V'(t)\, Z(M\mathbf{v}) \tag{5.11c}$$

Taking into account equations (5.5c), (5.11a) we have

$$V'(t)\{f(\mathbf{p})\} = \{\eta(\mathbf{p}, t) f(\mathbf{p})\} \tag{5.12}$$

where $\eta(\mathbf{p}, t)$ is a linear operator on $\mathfrak{H}^j$. Substituting (5.12) into (5.8), (5.11b) and (5.11c) we obtain respectively

$$\eta(\mathbf{p}, t_1)\,\eta(\mathbf{p}, t_2) = \eta(\mathbf{p}, t_1 + t_2) \tag{5.13a}$$

$$\eta(R\mathbf{p}, t) = D^j(R)\,\eta(\mathbf{p}, t)\, D^j(R^{-1}) \tag{5.13b}$$

$$e^{-i\mathbf{v}.\mathbf{p}t}\,\eta(\mathbf{p} - M\mathbf{v}, t) = e^{-i1/2Mv^2t}\,\eta(\mathbf{p}, t) \tag{5.13c}$$

for $\mathbf{p} \in R_3$ almost everywhere. From equation (5.13c) one has that

$$\eta(\mathbf{p}, t) = e^{-i1/(2M)p^2t}\,\vartheta(t)$$

where $\vartheta(t)$ is a linear operator on $\mathfrak{H}^j$; then by (5.13b), the irreducibility of $\mathscr{D}^j$ and (5.13a) one obtains

$$\eta(\mathbf{p}, t) = \exp\left[-i\left(\frac{1}{2M}p^2 + \lambda\right)t\right] I^{(j)} \tag{5.14}$$

where λ is a complex number. Taking into account (5.10), (5.7), (5.12) and (5.14) one has

$$V(t)\{f(\mathbf{p})\} = \left\{\exp\left[-i\left(\frac{1}{2M}p^2 + \lambda\right)t\right] f(\mathbf{p})\right\} \tag{5.15}$$

writing $\lambda = U - i(\gamma/2)$ one has $\gamma \geqslant 0$ due to the contractive character of $V(t)$ for $t \geqslant 0$; U can be interpreted as internal energy of the particle and γ as the inverse lifetime.

Finally let us require that the 'position' observable introduced by (5.1) is covariant, i.e.

$$\begin{aligned} &U(\mathbf{0}, \mathbf{v}, I; t)\, P(E)\, U^{-1}(\mathbf{0}, \mathbf{v}, I; t) = P(g(x, t)E) \\ &g(\mathbf{v}, t)\, E \equiv \{\mathbf{x} : \mathbf{x} - \mathbf{v}t \in E\} \end{aligned} \tag{5.16}$$

Equation (5.16) for $t = 0$ gives

$$G(\mathbf{v})\, P(E)\, G^{-1}(\mathbf{v}) = P(E) \tag{5.17}$$

which in turn implies (5.16) by (3.8) and (5.1).

Substituting (5.6c) into (5.17) and taking into account (5.5a) and (5.5b) by the irreducibility of the set of operators $Z(\mathbf{d})$, $T(\mathbf{a})$, $O(R)$, one has

$$S = I \tag{5.18}$$

so that by (5.6c)

$$G(\mathbf{v}) = Z(M\mathbf{v}) \tag{5.19}$$

In conclusion we have shown that an unstable Galilean particle is associated to an irreducible projective representation of the Galilei semigroup (unitary with exclusion of time translations), characterised by M, j, $\lambda = U_{-i\gamma/2}$.

The operators $V(t)$, $T(\mathbf{a})$, $O(R)$, $G(\mathbf{v})$ are given respectively by equations (5.15), (5.5c); (5.5b), (5.19) with (5.5a).

Correspondingly one has the obvious generalisation of the Schrödinger equation

$$i\frac{\partial\psi_{s_z}(\mathbf{x},t)}{\partial t} = \left(\frac{-1}{2M}\Delta_2 + U_{-i\gamma/2}\right)\psi_{s_z}(\mathbf{x},t) \tag{5.20}$$

for

$$\psi_{s_z}(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}}(u_{s_z}, f_{\gamma_0}(\mathbf{p},t))$$

where

$$\{f_{\gamma_0}(\mathbf{p},t)\} = V(t) f_{\gamma_0} \qquad \text{and} \qquad \{u_{s_z}\}_{s_z=-j,\,-j+1,\,\ldots,\,+j}$$

is the orthonormal set of eigenstates of $D^j(R)$, R being any rotation around the z-axis.

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